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ABSTRACT. Analysis of thermal stresses in an infinite, elastic, orthotropic cylinder subjected to axisymmetric heating. Exact formulas are derived for constant elastic parameters and for elastic parameters varying along the radius. A computational algorithm is given for a laminated cylinder with an arbitrary number of layers. A system of first-order differential equations suitable for numerical integration is obtained.

Consider an elastic infinitely long cylinder which is orthotropic and /24*
has a cylindrical anisotropy. We shall use a cylindrical coordinate system in which the z axis, directed along the axis of the cylinder, is at the same time the axis of anisotropy. The generalized Hooke law for this case has the form [1]

$$\left.\begin{aligned}
\varepsilon_{rr} &= \frac{1}{E_r}\sigma_{rr} - \frac{\nu_{r\varphi}}{E_\varphi}\sigma_{\varphi\varphi} - \frac{\nu_{rz}}{E_z}\sigma_{zz} + \alpha_r T,\\
\varepsilon_{\varphi\varphi} &= -\frac{\nu_{\varphi r}}{E_r}\sigma_{rr} + \frac{1}{E_\varphi}\sigma_{\varphi\varphi} - \frac{\nu_{\varphi z}}{E_z}\sigma_{zz} + \alpha_\varphi T,\\
\varepsilon_{zz} &= -\frac{\nu_{zr}}{E_r}\sigma_{rr} - \frac{\nu_{z\varphi}}{E_\varphi}\sigma_{\varphi\varphi} + \frac{1}{E_z}\sigma_{zz} + \alpha_z T,\\
\varepsilon_{\varphi z} &= \frac{1}{G_{\varphi z}}\sigma_{\varphi z},\quad \varepsilon_{rz} = \frac{1}{G_{rz}}\sigma_{rz},\\
\varepsilon_{r\varphi} &= \frac{1}{G_{r\varphi}}\sigma_{r\varphi}.
\end{aligned}\right| \qquad (1)$$

Here ε_{ij} and σ_{ij} are components of the strain and stress tensors; T is the temperature field; E, ν, G, α are the elasticity moduli, Poisson's ratios, shear moduli, and coefficients of linear expansion in the directions indicated by subscripts. Moreover,

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*Numbers in the margin indicate pagination in the original foreign text.

$$\frac{\nu_{r\varphi}}{E_\varphi}=\frac{\nu_{\varphi r}}{E_r},\quad \frac{\nu_{\varphi z}}{E_z}=\frac{\nu_{z\varphi}}{E_\varphi},\quad \frac{\nu_{rz}}{E_z}=\frac{\nu_{zr}}{E_r}. \tag{2}$$

All thermoelastic coefficients and the temperature field depend only on r. Considering the symmetry of our problem, we shall express the components of the strain tensor in terms of the radial and axial displacements, u(r) and w(z):

$$\begin{gathered}\varepsilon_{rr}=\frac{du}{dr},\qquad \varepsilon_{\varphi\varphi}=\frac{u}{r},\quad \varepsilon_{zz}=\frac{dw}{dz}=e=\text{const},\\ \varepsilon_{r\varphi}=\varepsilon_{rz}=\varepsilon_{\varphi z}=0.\end{gathered} \tag{3}$$

Using (1) and (3), we find the relationship between the stress tensor components and u(r):

$$\sigma_{rr}=A_{11}\frac{du}{dr}+A_{12}\frac{u}{r}+A_{13}e-\beta_r T, \tag{4}$$

$$\sigma_{\varphi\varphi}=A_{12}\frac{du}{dr}+A_{22}\frac{u}{r}+A_{23}e-\beta_\varphi T, \tag{5}$$

$$\sigma_{zz}=A_{13}\frac{du}{dr}+A_{23}\frac{u}{r}+A_{33}e-\beta_z T, \tag{6}$$ /25

$$\sigma_{r\varphi}=\sigma_{rz}=\sigma_{\varphi z}=0, \tag{7}$$

where

$$\begin{gathered}A_{11}=\frac{E_r(1-\nu_{\varphi z}\varphi_{z\varphi})}{D},\quad A_{12}=\frac{E_r(\nu_{r\varphi}+\nu_{rz}\nu_{z\varphi})}{D},\\ A_{13}=\frac{E_r(\nu_{rz}+\nu_{r\varphi}\nu_{\varphi z})}{D},\quad A_{22}=\frac{E_\varphi(1-\nu_{rz}\nu_{zr})}{D},\\ A_{23}=\frac{E_\varphi(\nu_{\varphi z}+\nu_{\varphi r}\nu_{rz})}{D},\quad A_{33}=\frac{E_z(1-\nu_{\varphi r}\nu_{r\varphi})}{D},\\ D=1-2\nu_{r\varphi}\nu_{\varphi z}\nu_{zr}-\nu_{rz}\nu_{zr}-\nu_{z\varphi}\nu_{\varphi z}-\nu_{r\varphi}\nu_{\varphi r},\\ \beta_r=A_{11}\alpha_r+A_{12}\alpha_\varphi+A_{13}\alpha_z,\\ \beta_\varphi=A_{12}\alpha_r+A_{22}\alpha_\varphi+A_{23}\alpha_z,\\ \beta_z=A_{13}\alpha_r+A_{23}\alpha_\varphi+A_{33}\alpha_z.\end{gathered}$$

Substituting (4) and (5) in the equation of equilibrium

$$\frac{d\sigma_{rr}}{dr}+\frac{1}{r}(\sigma_{rr}-\sigma_{\varphi\varphi})=0, \tag{8}$$

we obtain an equation for u(r)

$$\begin{gathered}A_{11}\frac{d^2u}{dr^2}+\left(A_{11}+r\frac{dA_{11}}{dr}\right)\frac{1}{r}\frac{du}{dr}-\left(A_{22}-r\frac{dA_{12}}{dr}\right)\frac{u}{r^2}=\\ =\left(\frac{A_{23}-A_{13}}{r}-\frac{dA_{13}}{dr}\right)e+\frac{d}{dr}(\beta_r T)+\frac{1}{r}(\beta_r-\beta_\varphi)T.\end{gathered} \tag{9}$$

Upon introducing the variables

$$u_1 = u,$$
$$u_2 = \sigma_{rr} = A_{11}\frac{du}{dr} + A_{12}\frac{u}{r} + A_{13}e - \beta_r T, \qquad (10)$$

(9) reduces to two equations of first order

$$\frac{du_1}{dr} = -\frac{A_{12}}{A_{11}}\frac{u_1}{r} + \frac{u_2}{A_{11}} - \frac{A_{13}}{A_{11}}e + \frac{\beta_r}{A_{11}}T,$$
$$\frac{du_2}{dr} = \left(A_{22} - \frac{A_{12}^2}{A_{11}}\right)\frac{u_1}{r^2} + \left(\frac{A_{12}}{A_{11}} - 1\right)\frac{u_2}{r} + \left(A_{23} - \frac{A_{12}A_{13}}{A_{11}}\right)\frac{e}{r} +$$
$$+ \left(\frac{A_{12}}{A_{11}}\beta_r - \beta_\varphi\right)\frac{T}{r}. \qquad (11)$$

The boundary conditions for a hollow cylinder are written as

$$\text{a)}\quad u_2(\alpha) = 0, \quad \text{b)}\quad u_2(b) = 0, \qquad (12)$$

where a and b are the inner and outer radii, respectively. For a solid cylinder Condition (12, a) is replaced by the condition that u_2 be finite at $r = 0$. The constant e can be determined from the condition of equilibrium along the z axis

$$\int_a^b \sigma_{zz} 2\pi r dr = 0. \qquad (13)$$

System (11) is convenient in numerical integration since it does not contain derivatives of the functions E, ν, β, T, that could be given by tables. In addition, the variables u_1 and u_2 are continuous when the parameters change discontinuously [2].

Equation (9) does not have an analytic solution if the coefficients E, ν have an arbitrary dependence on r. The solution may be found if the coefficients are stepwise functions of r. Physically this means that the cylinder is subdivided by surfaces $r = r_i$ into concentric layers $r_i \leq r \leq r_{i+1}$, each of which has its coefficients $E^{(i)}$, $\nu^{(i)}$. Then the equations for displacements $u_i(r)$ in each layer can be written as /26

$$\frac{d^2u_i}{dr^2} + \frac{1}{r}\frac{du_i}{dr} - \frac{A_{22}^{(i)}}{A_{11}^{(i)}}\frac{u_i}{r^2} = \frac{A_{23}^{(i)} - A_{13}^{(i)}}{A_{11}^{(i)}}\frac{e}{r} + \Phi_i(r), \qquad (14)$$

where

$$\Phi_i(r) = \frac{1}{A_{11}^{(i)}}\frac{d}{dr}(\beta_r^{(i)}T) + \frac{\beta_r^{(i)} - \beta_\varphi^{(i)}}{A_{11}^{(i)}}\frac{T}{r}.$$

The subscript i shows that when computing a given quantity one should substitute the coefficients for the ith layer.

A general solution of (14) can be found by the method of variation of the arbitrary constants

$$u_i(r) = c_i r^{n_i} + d_i r^{-n_i} + A_i(r) + f_i(r) e, \tag{15}$$

where c_i, d_i are constants of integration

$$n_i = \sqrt{\frac{A_{22}^{(i)}}{A_{11}^{(i)}}} = \sqrt{\frac{E_\varphi^{(i)}}{E_r^{(i)}} \frac{1 - \nu_{rz}^{(i)} \nu_{zr}^{(i)}}{1 - \nu_{\varphi z}^{(i)} \nu_{z\varphi}^{(i)}}},$$

$$A_i(r) = \frac{r^{n_i}}{2n_i} \int_0^r \Phi_i r^{1-n_i} \, dr - \frac{r^{-n_i}}{2n_i} \int_0^r \Phi_i r^{1+n_i} \, dr,$$

$$f_i(r) = \frac{A_{23}^{(i)} - A_{13}^{(i)}}{A_{11}^{(i)}} \cdot \frac{r}{1 - n_i^2}.$$

It will be noted that $\frac{A_{22}^{(i)}}{A_{11}^{(i)}} > 0,$ since $\nu^{(i)} \leqslant \frac{1}{2}$.

Substituting (15) in (4), we get

$$\sigma_{rr}^{(i)}(r) = l_i c_i r^{n_i - 1} + m_i d_i r^{-n_i - 1} + B_i(r) + g_i e, \tag{16}$$

where

$$l_i = n_i A_{11}^{(i)} + A_{12}^{(i)}, \qquad m_i = -n_i A_{11}^{(i)} + A_{12}^{(i)},$$

$$B_i(r) = \frac{l_i r^{n_i - 1}}{2n_i} \int_0^r \Phi_i r^{1-n_i} \, dr - \frac{m_i r^{-n_i - 1}}{2n_i} \int_0^r \Phi_i r^{1+n_i} \, dr - \beta_r^{(i)} T,$$

$$g_i = \frac{(A_{11}^{(i)} + A_{12}^{(i)})(A_{23}^{(i)} - A_{13}^{(i)})}{A_{11}^{(i)}(1 - n_i^2)} + A_{13}^{(i)}.$$

Letting $r = r_i$ in (15) and (16), we express the constants of integration c_i, d_i in terms of $u_i(r_i)$ and $\sigma_{rr}^{(i)}(r_i)$:

$$\left.\begin{aligned} c_i &= r_i^{-n_i}\frac{m_i u_i(r_i) - r_i\sigma_{rr}^{(i)}(r_i) - m_i A_i(r_i) + r_i B_i(r_i) + [r_i g_i - m_i f_i(r_i)]\,e}{m_i - l_i}, \\ d_i &= r_i^{n_i}\frac{-l_i u_i(r_i) + r_i\sigma_{rr}^{(i)}(r_i) + l_i A_i(r_i) - r_i B_i(r_i) + [l_i f_i(r_i) - r_i g_i]\,e}{m_i - l_i}. \end{aligned}\right| \tag{17}$$

Now, substituting (17) in (15) and (16), we obtain an expression for /27
displacements and radial stresses, which depends on the values of these quantities at the boundary of a given layer $r = r_i$:

$$\left.\begin{aligned} u_i(r) &= c_{11}^{(i)}(r)\,u_i(r_i) + c_{12}^{(i)}(r)\,\sigma_{rr}^{(i)}(r_i) + c_{13}^{(i)}(r)\,e + D_1^{(i)}(r), \\ \sigma_{rr}^{(i)}(r) &= c_{21}^{(i)}(r)\,u_i(r_i) + c_{22}^{(i)}(r)\,\sigma_{rr}^{(i)}(r_i) + c_{23}^{(i)}(r)\,e + D_2^{(i)}(r), \end{aligned}\right| \tag{18}$$

where

$$\left.\begin{aligned}
c_{11}^{(i)}(r) &= \frac{m_i\left(\frac{r}{r_i}\right)^{n_i} - l_i\left(\frac{r}{r_i}\right)^{-n_i}}{m_i - l_i}, \quad c_{12}^{(i)}(r) = \frac{r_i\left[\left(\frac{r}{r_i}\right)^{-n_i} - \left(\frac{r}{r_i}\right)^{n_i}\right]}{m_i - l_i}, \\
c_{13}^{(i)}(r) &= \frac{\left[l_i\left(\frac{r}{r_i}\right)^{-n_i} - m_i\left(\frac{r}{r_i}\right)^{n_i}\right] f_i(r_i) + r_i\left[\left(\frac{r}{r_i}\right)^{n_i} - \left(\frac{r}{r_i}\right)^{-n_i}\right] g_i}{m_i - l_i} + f_i(r), \\
D_1^{(i)}(r) &= \frac{l_i\left(\frac{r}{r_i}\right)^{-n_i} - m_i\left(\frac{r}{r_i}\right)^{n_i}}{m_i - l_i}\,A_i(r_i) + A_i(r) + \\
&\quad + \frac{r_i\left[\left(\frac{r}{r_i}\right)^{n_i} - \left(\frac{r}{r_i}\right)^{-n_i}\right]}{m_i - l_i}\,B_i(r_i), \\
c_{21}^{(i)}(r) &= \frac{m_i l_i\left[\left(\frac{r}{r_i}\right)^{n_i} - \left(\frac{r}{r_i}\right)^{-n_i}\right]}{(m_i - l_i)\,r}, \quad c_{22}^{(i)}(r) = \frac{m_i\left(\frac{r}{r_i}\right)^{-n_i} - l_i\left(\frac{r}{r_i}\right)^{n_i}}{m_i - l_i}, \\
c_{23}^{(i)}(r) &= \frac{m_i l_i\frac{1}{r}\left[\left(\frac{r}{r_i}\right)^{-n_i} - \left(\frac{r}{r_i}\right)^{n_i}\right] f_i(r_i) + \left[l_i\left(\frac{r}{r_i}\right)^{n_i} - m_i\left(\frac{r}{r_i}\right)^{-n_i}\right] g_i}{m_i - l_i} + g_i, \\
D_2^{(i)}(r) &= \frac{m_i l_i}{m_i - l_i}\,\frac{1}{r}\left[\left(\frac{r}{r_i}\right)^{-n_i} - \left(\frac{r}{r_i}\right)^{n_i}\right] A_i(r_i) + \\
&\quad + \frac{l_i\left(\frac{r}{r_i}\right)^{n_i} - m_i\left(\frac{r}{r_i}\right)^{-n_i}}{m_i - l_i}\,B_i(r_i) + B_i(r).
\end{aligned}\right|$$

$u_i(r_i)$ and $\sigma_{rr}^{(i)}(r_i)$ enter Equation (18) as arbitrary constants whose values are determined from the boundary conditions.

Letting $r = r_i$ in (18) and using the condition of continuity of displacements and radial stresses across a layer interface

$$u_{i+1}(r_{i+1}) = u_i(r_{i+1}), \quad \sigma_{rr}^{(i+1)}(r_{i+1}) = \sigma_{rr}^{(i)}(r_{i+1}), \tag{19}$$

we obtain a relationship between the constants for the ith and (i + 1)th layers:

$$\begin{aligned} u_{i+1}(r_{i+1}) &= c_{11}^{(i)}(r_{i+1})\,u_i(r_i) + c_{12}^{(i)}(r_{i+1})\,\sigma_{rr}^{(i)}(r_i) + c_{13}^{(i)}(r_{i+1})\,e + D_1^{(i)}(r_{i+1}), \\ \sigma_{rr}^{(i+1)}(r_{i+1}) &= c_{21}^{(i)}(r_{i+1})\,u_i(r_i) + c_{22}^{(i)}(r_{i+1})\,\sigma_{rr}^{(i)}(r_i) + c_{23}^{(i)}(r_{i+1})\,e + D_2^{(i)}(r_{i+1}). \end{aligned} \tag{20}$$

We shall now give an algorithm for finding the constants $u_i(r_i)$ and $\sigma_{rr}^{(i)}(r_i)$. Substituting (18) in (6), we find

$$\sigma_{zz}^{(i)}(r) = c_{31}^{(i)}(r)\,u_i(r_i) + c_{32}^{(i)}(r)\,\sigma_{rr}^{(i)}(r_i) + c_{33}^{(i)}(r)\,e + D_3^{(i)}(r), \tag{21}$$

where

$$\begin{aligned} c_{31}^{(i)}(r) &= A_{13}^{(i)}\frac{dc_{11}^{(i)}}{dr} + A_{23}^{(i)}\frac{c_{11}^{(i)}}{r}, \quad c_{32}^{(i)}(r) = A_{13}^{(i)}\frac{dc_{12}^{(i)}}{dr} + A_{23}^{(i)}\frac{c_{12}^{(i)}}{r}, \\ c_{33}^{(i)}(r) &= A_{13}^{(i)}\frac{dc_{13}^{(i)}}{dr} + A_{23}^{(i)}\frac{c_{13}^{(i)}}{r} + A_{33}^{(i)}, \quad D_3^{(i)}(r) = A_{13}^{(i)}\frac{dD_1^{(i)}}{dr} + A_{23}^{(i)}\frac{D_1^{(i)}}{r} - \beta_2^{(i)}T. \end{aligned} \tag{22}$$

A force N acts in the cross section, and its magnitude is given by /28

$$\begin{aligned} N &= \sum_i \int_{r_i}^{r_{i+1}} \sigma_{zz}^{(i)}(r)\,2\pi r\,dr = \sum_i u_i(r_i) \int_{r_i}^{r_{i+1}} c_{31}^{(i)}\,2\pi r\,dr + \\ &+ \sum_i \sigma_{rr}^{(i)}(r_i) \int_{r_i}^{r_{i+1}} c_{32}^{(i)}\,2\pi r\,dr + e\sum_i \int_{r_i}^{r_{i+1}} c_{33}^{(i)}\,2\pi r\,dr + \sum_i \int_{r_i}^{r_{i+1}} D_3^{(i)}\,2\pi r\,dr. \end{aligned} \tag{22}$$

On the inner surface $r = \alpha$ we have $\sigma_{rr}^{(0)}(\alpha) = 0$. Equations (18), (20), (22) imply that N and $\sigma_{rr}(b)$ are linear functions of $u_0(\alpha)$ and e

$$\begin{aligned} N(u_0(a), e) &= p_1 + q_1 u_0(a) + t_1 e, \\ \sigma_{rr}(b, u_0(a), e) &= p_2 + q_2 u_0(a) + t_2 e. \end{aligned} \tag{23}$$

With the aid of (20) and (22) we find N (0,0), N (0, 1), N (1, 0), $\sigma_{rr}(b,0,0)$, $\sigma_{rr}(b,0,1)$, $\sigma_{rr}(b,1,0)$. The coefficients p, q, t in (23) can be simply expressed in terms of the former

$$\begin{aligned} p_1 &= N(0, 0), \quad q_1 = N(1, 0) - N(0, 0), \quad t_1 = N(0, 1) - N(0, 0), \\ p_2 &= \sigma_{rr}(b, 0, 0), \quad q_2 = \sigma_{rr}(b, 1, 0) - \sigma_{rr}(b, 0, 0), \\ t_2 &= \sigma_{rr}(b, 0, 1) - \sigma_{rr}(b, 0, 0). \end{aligned} \tag{24}$$

Substituting (24) in (23), and setting N and $\sigma_{rr}(b)$ equal to zero, we obtain two equations from which we can determine $u_0(\alpha)$ and e:

$$\left.\begin{array}{c} N(0,0)+[N(1,0)-N(0,0)]\,u_0(a)+[N(0,1)-N(0,0)]\,e=0, \\ \sigma_{rr}(b,0,0)+[\sigma_{rr}(b,1,0)-\sigma_{rr}(b,0,0)]\,u_0(a)+ \\ +[\sigma_{rr}(b,0,1)-\sigma_{rr}(b,0,0)]\,e=0. \end{array}\right| \quad (25)$$

Knowing $u_0(\alpha)$ and e, with the aid of (20), we find all $u_i(r_i)$ and $\sigma_{rr}^{(i)}(r_i)$.

In the case of a solid cylinder we introduce a certain small radius $\alpha \ll b$ and require that $u_0(\alpha)$. Then the algorithm remains the same if in Equations (23) - (25) we replace $u_0(\alpha)$ with $\sigma_{rr}^{(0)}(\alpha)$.

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